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Aggregation of utility and social choice: A topological characterization

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Abstract

We study the topological properties of aggregation maps combining individuals' preferences over n alternatives, with preference expressed by a real-valued, n-dimensional utility vector $\mathbf{u}$ defined on an interval scale. Since any such utility vector is specified only up to arbitrary affine transformations, the space of utility vectors $\mathbb{R}^n$ may be partitioned into equivalence classes of the form $\{a\mathbf{u} + b\mathbf{1} \mid a \in \mathbb{R}_0^+,\ b \in \mathbb{R}\}$. The quotient space, denoted T, is shown to be the union of the $(n-2)$-dimensional sphere denoted S with the singleton $\{0\}$, which corresponds to indifference or null preference. The topology of T is non-Hausdorff, placing it outside the scope of most existing theory (e.g., J. Econom. Theory 31 (1983) 68–87.). We then investigate the existence and nature of continuous aggregation maps under the four scenarios of allowing or disallowing null preference both in individual and in social choice, i.e. maps $f : P \times \cdots \times P \to Q$ with $P, Q \in \{T, S\}$. We show that there exist continuous, anonymous, unanimous aggregation maps iff the outcome space includes the null point (i.e., $Q = T$), and provide a simple well-behaved example for the case $f : S \times \cdots \times S \to T$. Similar examples exist for $f : T \times \cdots \times T \to T$, but these and all other maps have a property of always either over- or under-allocating influence to each voter (in a specific manner). We conclude that there exist acceptable aggregation rules if and only if null preference is allowed for the society but not for the individual.

Keywords: Social choice; Topological social choice; Utility; Cardinal preference; Aggregation; Indifference; Impossibility theorem

1. Introduction

The field of social choice theory is concerned with methods for aggregating the preferences of individuals in a population into a social preference or outcome. One common example of such a problem is popular elections, in which voters' preferences over candidates or political parties, given as favorites, approved subsets, rankings, or scores, are used to determine the winner or the relative power of the contenders. The field has been largely motivated by impossibility results showing that certain normatively desirable characteristics of the procedure for preference aggregation turn out to be incompatible. For instance Arrow (1963) showed that, whenever there are at least three alternatives (candidates), the only aggregation rules simultaneously satisfying independence of irrelevant alternatives (IR) and the Pareto principle are simple dictatorships. The mathematical characterization of such impossibilities has since been evolved from Arrow's combinatorial approach to a topological one where the space of preferences is endowed with a topological structure (see Lauwers, 2000 for a thorough review).

In topological choice theory, the additional structural information associated with the set of preferences enriches the mathematical content of the problem and allows for definition of further desirable properties of aggregation rules. For example, one often considers the continuity of aggregation maps, which corresponds to graceful dependence of the aggregated outcome on the preferences of individuals, which are often assumed to be noisy or imperfectly measured. This continuity property plays a similar role to that of the IR axiom, as a consistency requirement among the outcomes

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associated with different preference profiles (Lauwers, 2000). A further concept provided by the topological approach is that of homotopy (continuous deformation) between maps, which is a natural analog to continuous social change (i.e., change of the voting system).

Following in the Arrovian spirit, the central focus in the field of topological social choice is the question of when there exist continuous aggregation functions that satisfy certain desirable axioms. The most commonly investigated axioms, respective analogs to non-dictatorship and Pareto, are anonymity and respect of unanimity (to be defined below). In a now classic theorem, Chichilnisky and Heal (1983; see also Chichilnisky, 1980, 1982a, 1996) showed that under certain technical assumptions there exist continuous aggregation functions with these two properties if and only if the topological space of preferences is contractible. A contractible space is one that can be continuously deformed through itself into a point; intuitively it has no 'holes'. Baryshnikov (1997) has since made progress in unifying the topological and combinatorial approaches by deriving Arrow's (1963) theorem as a corollary to that of Chichilnisky and Heal (1983), as a consequence of the non-contractibility of the space of total orders of a finite set of alternatives under a suitable topology.

In characterizing aggregation rules, social choice theories have focused either on ordinal preferences, i.e. without information regarding the magnitude or intensity of preference for any alternative over another, or on cardinal preferences, in which the utility associated with each alternative is taken into account. In the latter case, one standard approach is to represent individual preferences by real-valued functions over the alternatives, either deterministically (e.g., Harsanyi, 1955; Selinger, 1986; Coulhon & Mongin, 1989) or probabilistically (Marley, 1992). For instance Coulhon and Mongin (1989), in an extension of Harsanyi's (1955) original theorem, show that under this framework any aggregation function satisfying the Pareto principle and a strong version of IR must take the form of a linear combination over individuals' utilities. However, one drawback of such an approach is that it fails to take into account the affine invariance property of utility. According to the axiomatization of von Neumann and Morgenstern (1944), an individual's utility function is only inferrable from his or her preferences among pairs of lotteries. Because such choice behavior is invariant under any positive linear transformation of the utility function, utility is properly defined on an interval scale. That is, for any utility vector $\mathbf{u}$ over n alternatives, and any $a \in \mathbb{R}_0^+$ (the strictly positive reals) and $b \in \mathbb{R}$, the utilities represented by $\mathbf{u}$ and by $a\mathbf{u} + b\mathbf{1}$ are one and the same (here $\mathbf{1}$ represents the constant vector $[1, \dots, 1]$). This defines an equivalence relation among utility functions, and significantly changes the structure of the space of underlying utilities. This equivalence relation renders meaningless such axioms as Coulhon and Mongin's (1989) version of IR, which states that the societal utility of a particular outcome is dependent only on the individuals' utilities for that outcome. Proper treatment of cardinal preference aggregation must take the equivalence relation inherent in utilities into account, either by requiring the aggregation function to be invariant under positive linear transformations of the inputs (D'Aspremont & Gevers, 1977) or by defining preferences to be equivalence classes rather than individual elements of $\mathbb{R}^n$ (Kalai & Schmeidler, 1977; Chichilnisky, 1985).

Just as it is imperative to properly consider the measurement scale of utility functions, it is also crucial to carefully consider the role of null preference, i.e. total indifference among the alternatives. In their foundational work, Chichilnisky (1980) and Chichilnisky and Heal (1983) explicitly ruled out the null preference from consideration. Chichilnisky (1982a) treats the case of vanishing ordinal preferences, but only as an isolated component of the preference space. This topology, which we will argue is an inadequate model of preferences, in effect reduces the problem to the case where the null point is excluded. In her analysis of cardinal preference aggregation, Chichilnisky (1985) follows the same approach taken here, and considers utilities to be defined as equivalence classes of real-valued functions under positive linear translations, with null preference corresponding to the class of constant functions. As in her treatment of ordinal utility (Chichilnisky, 1982a), the topology Chichilnisky (1985) derives for these equivalence classes is disconnected, with the null point an isolated component. Under this topology, the resolution theorem of Chichilnisky and Heal (1983) implies that there do not exist continuous, anonymous, unanimous aggregation functions for the space of cardinal preferences. However, as we argue below, Chichilnisky's (1985) topology for utilities is arbitrary and poorly motivated. We believe that the correspondence between these two separate attempts to topologize the null preference speaks to the complexity inherent in the concept of perfect indifference, as will be further illustrated by the subtlety of the topological results to follow.

In the present article, we revisit the question of topological aggregation of cardinal preferences (utilities), with specific attention to the role of the null preference. We begin by deriving the topology of the space of utilities over a finite set of alternatives, defined as a quotient space of the space of all real-valued functions over these alternatives. The topology that arises from our analysis is the same as one briefly considered by Le Breton and Uriarte (1990), and differs from that assumed in previous characterizations of both cardinal preferences (Chichilnisky, 1985) and ordinal preferences (Chichilnisky, 1982a), specifically in terms of the role of the null point. We then carry out a detailed analysis of the role of the null preference in determining the existence and nature of aggregation maps, by studying the types of maps that can arise as a function of whether indifference is allowed for individual preferences, social outcomes, or both. Our central result is a possibility result, complementary to previous impossibility theorems, stating that there exist continuous, anonymous aggregation maps that respect unanimity if and only if the null preference is allowed for the society. We further find that, in the case of null preference being allowed for both the society and individuals, all continuous aggregation maps exhibit the following pathology: For every voter x and every profile of preferences for the remaining voters, either x's preference has no effect on the aggregated outcome, or else a null preference for x causes the societal preference to be null as well. However, when null preference is allowed for the society but not for the individual, a simple example can be given of an aggregation map satisfying all of the desirable axioms we consider. We conclude that there exist acceptable aggregation rules only in this latter case.

2. Notation and background

Previous work (e.g., Chichilnisky, 1980, 1982a, 1982b, 1996; Chichilnisky & Heal, 1983), has considered topological spaces P representing the set of preferences applicable to a given choice situation. Though P is generally taken to be a space of orderings over some manifold of outcomes, much of the theory applies to an arbitrary topological space. The primary goal in most research of this type is to determine under what circumstances individual preferences can be aggregated in a manner satisfying a given set of axioms.

2.1. Definitions

We consider preferences among a fixed set of n alternatives. The topological space of individual preferences is denoted P. When the space of social preferences differs, it is denoted Q. A preference $p \in P$ (or Q) induces a weak ordering $\preccurlyeq_p$ on outcomes, where $x \preccurlyeq_p y$ indicates that y is at least as preferred as x under the preference p. We do not identify p with $\preccurlyeq_p$ because $\preccurlyeq_p$ may not contain all of the information in p, e.g., under some interpretations p also contains information about strength of preference. Null preference is denoted as O.

A *profile* $\mathbf{p} = (p_1, \ldots, p_k)$ of preferences for k individuals is an element of P^k, the k-fold Cartesian product of P.

Aggregation functions are defined as maps $f: P^k \to Q$, assigning to each profile of individual preferences a preference for the society as a whole.

An aggregation function f is *anonymous* iff it is invariant under arbitrary permutations of its arguments (voters). Thus for all bijections $\pi: \{1, \ldots, k\} \to \{1, \ldots, k\}$:

$$f(p_1, \ldots, p_k) = f(p_{\pi(1)}, \ldots, p_{\pi(k)}).$$

This condition is strictly stronger than Arrow's non-dictatorship requirement; whereas non-dictatorship still allows for strong imbalances in the roles of the voters, anonymity implies perfect symmetry.

An aggregation function f *respects unanimity* (or is unanimous) iff, whenever all voters have the same preference, f returns that preference; i.e., $\forall p \in P\, [f(p, \ldots, p) = p]$.

Chichilnisky functions are continuous aggregation maps that are anonymous and respect unanimity.

An additional axiom that is desirable to require of maps whose ranges include the null outcome is *efficiency*. If the domain is equipped with a measure μ then we can define an efficient aggregation map to be one for which $\mu(f^{-1}(0)) = 0$, that is the set of profiles sent to 0 has measure zero. Under reasonable assumptions about the preferences of the individual voters (e.g., that their a priori distribution is absolutely continuous with respect to μ), the probability of a null outcome under an efficient aggregation rule will be zero. In the subsequent analyses wherein we treat individual choice spaces P given by a sphere with the possible inclusion of a null point, the measure assumed on P^k is the product measure derived from the standard (Lebesgue) measure λ on the sphere, with $\lambda(\{0\}) \triangleq 0$. It should be noted though that the particular choice of measure is largely inconsequential, as any two measures that are absolutely continuous with respect to each other will yield the same efficiency requirement.

The *Pareto principle* states that f respects unanimity in any binary comparison. That is, for any pair of outcomes x, y:

$$\forall i [x \preccurlyeq_{p_i} y] \Rightarrow x \preccurlyeq_{f(\mathbf{p})} y.$$

In the case of linear preferences, where P reduces either to a sphere S^{n-2} or to $S^{n-2} \cup \{0\}$ (see below), the Pareto principle is equivalent to the following constraint:

$$\forall q \in S: \quad \forall i [p_i \cdot q \geqslant 0] \Rightarrow f(\mathbf{p}) \cdot q \geqslant 0,$$

where P is taken to be embedded in $\mathbb{R}^n$, as described below, for evaluation of inner products (denoted by "·"). It is straightforward to verify that this principle implies the respect of unanimity axiom.

Weak positive association (*WPA*) is a technical axiom that does not play a major role in the present analyses, but is relevant to past results (e.g., Chichilnisky, 1982b). An aggregation function f satisfies WPA iff for all profiles $\mathbf{p}$ satisfying $f(\mathbf{p}) = -p_i$ for some i, $f(-p_i, \ldots, -p_i, p_i, -p_i, \ldots, -p_i) \neq p_i$. Here, $-p_i$ represents the point antipodal to p_i, when $P = S$. Thus WPA states that whenever the outcome is exactly opposite the preference of one individual, a switch of everyone else's preference to that opposite outcome (with the ith voter remaining unchanged) cannot change the result to match the will of the given voter.

A *dictatorship* is a projection onto the preference of a single voter, i.e. an aggregation function f for which $\exists i\, \forall \mathbf{p} \in P^k [f(\mathbf{p}) = p_i]$.

As previously mentioned, homotopy between aggregation maps can be viewed as continuous social change, thus providing a kind of equivalence between maps (Chichilnisky, 1980). However, the concept of homotopy can be a quite broad one; for instance, if the image space is contractible then all maps are homotopic to one another. Baryshnikov (2000) refines the idea of homotopy to that of *isotopy*, which he defines as a homotopy preserving certain properties of the boundary maps. For instance, a Pareto-isotopy between Pareto maps is a homotopy for which all intermediate maps are Pareto as well. The restriction from homotopy to isotopy is desirable because it recognizes the constitutional constraints that are already being assumed for the end-maps in question.

2.2. Past results on aggregation

The question of when there exist Chichilnisky aggregation functions, i.e. maps that are continuous and anonymous and that respect unanimity, was answered by the resolution theorem of Chichilnisky and Heal (1983), so named because it provided a necessary and sufficient condition for when the social choice paradox can be resolved.

Resolution Theorem (Chichilnisky & Heal, 1983). *Assuming the preference space P is a parafinite CW complex,*[1] *there exist Chichilnisky functions $f : P^k \to P$ for all $k \geqslant \mathbb{N}$ if and only if P is contractible.*

A more direct analog to Arrow's (1963) impossibility theorem is the following:

Theorem (Chichilnisky, 1982b). *Any continuous aggregation function f that satisfies the WPA and Pareto conditions is homotopic to a dictatorship.*

Previous work by Chichilnisky (1980, 1982a) and Chichilnisky and Heal (1983) has considered as choice space all ordinal preferences over some convex manifold of outcomes, e.g. the m-dimensional ball B^m. Ordinal preferences are viewed as equivalence classes of real-valued continuous functions $\mathbf{u}$ on outcomes, with $\mathbf{u} \sim \mathbf{u}'$ iff $\forall x, y \in B^m [\mathbf{u}(x) \geqslant \mathbf{u}(y) \Leftrightarrow \mathbf{u}'(x) \geqslant \mathbf{u}'(y)]$ (Chichilnisky, 1980). This equivalence relation allows each preference to be represented by its corresponding set of isopreference classes within B^m. Under assumption of differentiability and non-satiation of the preference, these isopreference classes are co-dimension 1 submanifolds, and thus they give a foliation of the outcome space that can be represented by its unit normal vector field (Chichilnisky, 1980). In other words, each preference can be represented by the vector field of its normalized gradient (the non-satiation assumption implies the gradient is everywhere non-zero). When preferences are restricted to be linear, gradients are constant, and P is equal to the normalized (non-zero) tangent space of B^m at any one point, i.e. the sphere S^{m-1}. Because the sphere is not contractible, there exist no Chichilnisky aggregation rules by the resolution theorem.

2.3. Past results on topologizing the null preference

Chichilnisky (1982a) extends the above treatment to include vanishing preference gradients. In this case the set of ordinal preferences has a local structure (at all points of the outcome manifold) given by $S^{m-1} \cup \{0\}$. The question of the existence of Chichilnisky functions on this preference space thus reduces (after some careful logic) to that of Chichilnisky functions $f : (S^{m-1} \cup \{0\})^k \to S^{m-1} \cup \{0\}$. Under the closed convergence topology used by Chichilnisky (1982a), $S^{m-1} \cup \{0\}$ inherits the Euclidean topology from its embedding in $\mathbb{R}^m$, and thus $\{0\}$ is an open set. Under this topology the outcome space is disconnected, which leads to little flexibility of the aggregation function; one can simply consider the restriction of the map to the connected component $(S^{m-1})^k$, infer by respect of unanimity that its image must lie in S^{m-1}, and derive the impossibility result in the standard fashion (Chichilnisky, 1982a). However, Le Breton and Uriarte (1990) criticize the closed convergence topology as unnatural, and state that under an alternative topology, in which $\{0\}$ is not an open set, the impossibility result does not hold.

Another treatment of topological preference aggregation that addresses the null preference is Chichilnisky's (1985) analysis of cardinal (utility-based) preferences. As described above, such preferences are defined as equivalence classes of $\mathbb{R}^n$, where the elements $\mathbf{u} \in \mathbb{R}^n$ represent real-valued functions over a finite set of alternatives. The equivalence relation is induced by the set of all positive linear transformations, $\{\mathbf{u} \mapsto a\mathbf{u} + b\mathbf{1} \mid a \in \mathbb{R}_0^+, b \in \mathbb{R}\}$. In evaluating the topological space of equivalence classes, Chichilnisky (1985) follows Kalai and Schmeidler (1977) in normalizing every utility

[1] A CW complex is an extension of the notion of a simplicial complex, and is composed of simplices of possibly unbounded dimensionalities. See Chichilnisky and Heal (1983) or Maunder (1970) for a complete definition.

function to have least and greatest components equal to 0 and 1, respectively. Those functions $b\mathbf{1}$ that give equal value to all alternatives are an exception and are mapped to the constant function $\mathbf{0}$. The set of equivalence classes (i.e., the set of distinguishable utilities) can now be represented by a subset of $\mathbb{R}^n$ that contains exactly one member of each class:

$$\{\mathbf{u}\in\mathbb{R}^n \mid \forall i\,[u_i\in[0,1]]\ \&\ \exists j,k\,[u_j=0,u_k=1]\}\cup\{\mathbf{0}\}.$$

As a subset of $\mathbb{R}^n$, this space is homeomorphic to $S^{n-2}\cup\{0\}$, with the same topology as considered in Chichilnisky (1982a). Thus Chichilnisky (1985) concludes that there do not exist adequate aggregation functions for the space of cardinal preferences over a finite set of outcomes.

In our view, Chichilnisky's (1985) and Kalai and Schmeidler's (1977)[2] error was in inferring that the (quotient) topology of the equivalence classes is the same as that inherited by the embedding of the above representative set in the original space. In general the latter topology is in fact larger, i.e. it has more open (and closed) sets, and can depend on the choice of representative set. For example, the equivalence classes of utilities can also be represented by:

$$\{\mathbf{u}\in\mathbb{R}^n \mid \exists j,k\,[u_j=0,u_k=k,\ \forall i<k\,[u_i\in[0,k]],\ \forall i>k\,[u_i\in[0,k)]]]\}\cup\{\mathbf{0}\}.$$

This representation can be achieved from the earlier one by multiplying each non-zero element $\mathbf{u}$ by $\max\{k \mid u_k=1\}$. As a subset of $\mathbb{R}^n$ this space is a disjoint union of k $(n-2)$-dimensional partially closed disks and the singleton $\{\mathbf{0}\}$. As all components are contractible, the space admits Chichilnisky aggregators. Thus, it is clear that the representative-set approach gives inconsistent results, and that in order to determine the proper topological structure on the space of utilities we must derive the quotient topology directly. We turn now to such a derivation.

3. Topological characterization of utility

3.1. Utility as cardinal preference

In general, the topological space of n-ary observations on a given measurement scale can be derived as a quotient space of $\mathbb{R}^n$, with equivalence defined by the characteristic automorphism group of the measurement scale. That is, for any automorphism group G of the reals, we can define an equivalence relation on $\mathbb{R}^n$ by

$$\mathbf{x}\sim_G\mathbf{y}\Leftrightarrow\exists g\in G,\ \forall i\,[g(x_i)=y_i].$$

If this condition is satisfied, then $\mathbf{x}$ and $\mathbf{y}$ are indistinguishable as n-ary observations under the measurement scale defined by G. Thus the space of n-ary observations is given by $\mathbb{R}^n/\sim_G$.

In applying this analysis to the interval scale, the relevant automorphism group is the set of affine transformations $I=\{\mathbf{u}\mapsto a\mathbf{u}+b\mathbf{1} \mid a\in\mathbb{R}_0^+, b\in\mathbb{R}\}$. Thus we have $\mathbf{u}\sim_I a\mathbf{u}+b\mathbf{1}$ for all $a\in\mathbb{R}_0^+, b\in\mathbb{R}$ and $\mathbf{u}\in\mathbb{R}^n$.

In order to see how $\mathbb{R}^n$ collapses under the equivalence relation $\sim_I$, we first write I as the product (under functional composition) of two subgroups:

$$I=\{\mathbf{u}\mapsto a\mathbf{u} \mid a\in\mathbb{R}_0^+\}*\{\mathbf{u}\mapsto\mathbf{u}+b\mathbf{1} \mid b\in\mathbb{R}\}.$$

The first of these subgroups, corresponding to scalar invariance, defines the ratio scale. The second corresponds to the additional translational invariance present in the interval scale. The equivalence classes under each of these two subgroups for the case $n=3$ are shown graphically in Figs. 1A and B. The scalar subgroup equates points lying on any ray from the origin, whereas the translation subgroup equates points differing by any multiple of the diagonal vector $\mathbf{1}$. When the full automorphism group is considered, these two sets of equivalence classes "merge" to yield the partition shown in Fig. 1C. As is evident by the diagram, the structure of these equivalence classes is that of a circle S^1 with an additional point to be denoted 0 (Fig. 1D). The 0 point corresponds to the diagonal, i.e. the equivalence class of functions that rate all three alternatives equally. (Note that the operations of scalar multiplication and translation are redundant for this unique unidimensional class.)

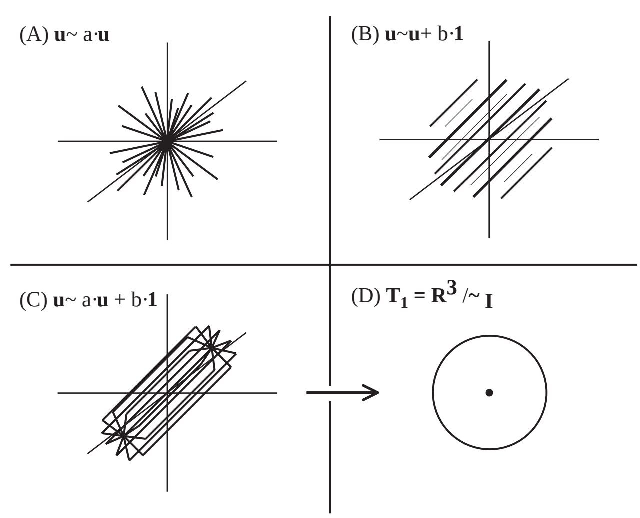

Fig. 1. Equivalence classes in $\mathbb{R}^3$ under linear transformations. (A) Equivalence under scalar multiplication, which gives the ratio scale of measurement. (B) Equivalence under constant-vector (diagonal) translation. (C) Equivalence under the full group I of linear transformations, defining the interval scale. (D) The quotient space $T^1=\mathbb{R}^3/\sim_I$.

[2] In the proof of their central theorem, Kalai and Schmeidler (1977) only use their topology to imply that certain sequences of profiles converge (see their Lemma 3). Since weakening the topology of a space can only increase the class of convergent sequences, their result still holds in the topology we derive here.

We refer to this space, endowed with the quotient topology inherited from $\mathbb{R}^3$, as T^1.

A crucial observation regarding the topology of T^1 concerns the point $0 = \langle \mathbf{0} \rangle$, and can be illustrated with reference to the equivalence class partition of $\mathbb{R}^3$ in Fig. 1C. Given an arbitrary point $b\mathbf{1} \in \langle \mathbf{0} \rangle$, any open neighborhood of $b\mathbf{1}$ will intersect all other equivalence classes. In this way, $\langle \mathbf{0} \rangle$ can be seen to be "arbitrarily close" to every other class. However, the reverse statement is false; in fact, any member of an equivalence class other than $\langle \mathbf{0} \rangle$ will be uniformly bounded away from all elements of $\langle \mathbf{0} \rangle$. In the topology of T^1, these observations translate to the fact that $\{0\}$ is closed but not open, and in fact the only open neighborhood of 0 is the entire space T^1. As we show next, this fact generalizes to T^m for all m.

Proposition 1 (Characterization of utility space). *The topological space of utilities over n alternatives, defined as* $\mathbb{R}^n/\sim_I$, *is homeomorphic to* $T^{n-2} \triangleq S^{n-2} \cup \{0\}$, *with topology in terms of open sets given by*

$$\mathscr{T}(T^{n-2}) = \mathscr{T}(S^{n-2}) \cup \{T^{n-2}\}.$$

That is, the open sets of T^{n-2} *are precisely the open sets of* S^{n-2}, *along with the entire space* T^{n-2}. *In particular,* S^{n-2} *is open in* T^{n-2} (*and hence* $\{0\}$ *is closed*), *but the only open neighborhood of* 0 *is* T^{n-2}.

Proof. Our first step in characterizing the quotient space will be to assign to each equivalence class a representative member. We will at times in this paper refer to certain utilities by their representatives, thereby considering the space of utilities as embedded in $\mathbb{R}^n$ (for instance for convenience in defining certain aggregation functions), although we will *not* equate the topology of utility space with the topology implied by this embedding.

Consider any $\mathbf{u} \in \mathbb{R}^n - \mathbb{R}\mathbf{1}$, that is with $u_i \neq u_j$ for some i and j. Define a representative $\hat{\mathbf{u}}$ of the equivalence class $\langle \mathbf{u} \rangle$ as follows: Define $b_\mathbf{u} = -\frac{1}{n}\sum u_i$ and $a_\mathbf{u} = (\sum (u_i + b_\mathbf{u})^2)^{-1/2}$, and let $\hat{\mathbf{u}} = a_\mathbf{u}\mathbf{u} + a_\mathbf{u} b_\mathbf{u}\mathbf{1}$. Clearly $\hat{\mathbf{u}} \sim_I \mathbf{u}$, so $\hat{\mathbf{u}} \in \langle \mathbf{u} \rangle$. Furthermore, this normalizing operation can be seen to be invariant under positive linear transformation, i.e. $\hat{\mathbf{u}} = \widehat{a\mathbf{u} + b\mathbf{1}}$ for any $a \in \mathbb{R}_0^+, b \in \mathbb{R}$. Therefore the mapping $\langle \mathbf{u} \rangle \mapsto \hat{\mathbf{u}}$ is well defined. Next observe that for all $\mathbf{u} \in \mathbb{R}^n - \mathbb{R}\mathbf{1}$, $\hat{\mathbf{u}}$ lies in the intersection of the hyperplane $\{\mathbf{v} \in \mathbb{R}^n \mid \sum v_i = 0\}$ with the unit sphere $\{\mathbf{v} \in \mathbb{R}^n \mid \sum v_i^2 = 1\}$. This intersection is an $(n-2)$-dimensional sphere which we identify with the abstract sphere S^{n-2}. Conversely, every member of this sphere is the representative for its equivalence class, since $\mathbf{u} \in S^{n-2}$ leads to $b_\mathbf{u} = 0$ and $a_\mathbf{u} = 1$, implying $\hat{\mathbf{u}} = \mathbf{u}$. Therefore the set of representatives $\{\hat{\mathbf{u}} \mid \mathbf{u} \in \mathbb{R}^n - \mathbb{R}\mathbf{1}\}$ corresponds precisely to the sphere S^{n-2}. If we now define $\widehat{b\mathbf{1}} = \mathbf{0}$ for all $b \in \mathbb{R}$, and identify the origin $\mathbf{0}$ with the null preference 0, we obtain a 1–1 correspondence between $S^{n-2} \cup$ and the elements of $\mathbb{R}^n/\sim_I$.

To complete the proof we must determine the topology of the quotient space. Quotient topology is defined as the weak topology induced by the quotient map Λ, which sends points in $\mathbb{R}^n$ to their equivalence classes in T^{n-2}. In other words, $A \subset T^{n-2}$ is open (closed) iff $\Lambda^{-1}(A)$ is open (closed) in $\mathbb{R}^n$. Consider first a closed set A in the quotient space T^{n-2}. Since $S^{n-2} \cup$ is closed with a subset of $\mathbb{R}^n$, so is $(S^{n-2} \cup) \cap \Lambda^{-1}(A) = A$. Thus the quotient topology is contained in the topology induced by the embedding. The open sets of this latter topology are B and $B \cup \{0\}$ for all B open in S^{n-2}. However, the sets $B \cup \{0\}$ for all $B \subsetneq S^{n-2}$ (including $B = \emptyset$) are not open in T^{n-2} by the following argument. Take $p \in S^{n-2} - B$. The pre-image $\Lambda^{-1}(B \cup \{0\})$ contains the point $\mathbf{0}$ but not ϵp for any $\epsilon > 0$. The distance from $\mathbf{0}$ to ϵp is equal to ϵ, i.e. $\mathbf{0}$ is arbitrarily close to points outside $\Lambda^{-1}(B \cup \{0\})$. Therefore $\Lambda^{-1}(B \cup \{0\})$ cannot be open.

We have now proven one direction of the claim: The topology given in the statement of the proposition contains the true topology of T^{n-2}. Since $\Lambda^{-1}(T^{n-2}) = \mathbb{R}^n$ is open, it remains to show that $\Lambda^{-1}(B)$ is open in $\mathbb{R}^n$ for all B open in S^{n-2}. It is sufficient to prove this statement for all elements of the sub-basis $\{\check{q} \mid q \in S^{n-2}\}$, where $\check{q} = \{p \in S^{n-2} \mid p \cdot q > 0\}$. For any q, $\Lambda^{-1}(\check{q}) = \{\mathbf{u} \in \mathbb{R}^n \mid \mathbf{u} \cdot q > 0\}$, which is an open half-space in $\mathbb{R}^n$. Thus the members of the sub-basis for the topology of S^{n-2} are all open in T^{n-2}. □

Two properties of the topology of T^{n-2} should be emphasized. First, T^{n-2} is non-Hausdorff: Because the only open neighborhood of 0 is all of T^{n-2}, 0 cannot be "separated" from any other point. Second, T^{n-2} is contractible, as can be seen from the contraction of $\mathbb{R}^n$ given by $\Phi_t(\mathbf{u}) = (1-t)\mathbf{u}$ for $t \in [0, 1]$. When projected along the quotient map Λ, Φ induces the following contraction of T^{n-2}:

$$\Psi_t(p) = \begin{cases} p & \text{for } t < 1, \\ 0 & \text{for } t = 1. \end{cases}$$

This contraction looks like a sudden jump of all points to 0, and is continuous because of the non-Hausdorff topology of T^{n-2}.

Because the contributions of this article depend crucially on the topological properties of indifference, some additional comments are in order at this point. The sole purpose of defining a topology on a preference space is to determine what qualifies as a continuous map (both for aggregation rules and for homotopies between them). The purpose of the continuity requirement is, in turn, to prevent hypersensitivities in the aggregation rule by ensuring that small changes in the inputs do not cause a large change in the output. This is the primary guiding principle behind topological social choice theory. Therefore, in evaluating the descriptive relevance of a particular topology, one must consider what is meant by a "small change." In the case of null preference, it should be apparent that an arbitrarily small change is sufficient to swing the voter's preference in any given direction. For example, an individual who was perfectly indifferent between alternatives A, B, and C could be induced to prefer A over the other two by offering to pay her as little as 1¢ if A were the outcome. Furthermore, probabilistically distributing the penny between A and B (i.e., setting some probability that B will bring the payoff instead of A) could achieve a ranking of A over B over C with the ratio of the strength of preference of A over B to that of B over C taking on any value desired. Because arbitrarily small perturbations of this type are able to induce a switch from indifference to any other preference, such switches in preference should be regarded as arbitrarily small as well. This reasoning is captured in the present quotient-space approach, and is formalized within the topology of T^{n-2}.

3.2. Utilities as ordinal preferences in a probability simplex

An alternative representation of the space of utilities over n alternatives comes from the definition of utility as ordinal preference over lotteries (von Neumann & Morgenstern, 1944). Specifically, consider the $(n-1)$-dimensional simplex M of probability distributions over the alternatives and define a utility as a preference over this space. The axioms of von Neumann and Morgenstern (1944) imply that these preferences must be linear. As in Chichilnisky (1980, 1982a), preferences can thus be identified with their normalized (constant) gradients, implying a correspondence between the space of utilities and $S^{n-2} \cup \{0\}$. This unifies the present treatment of cardinal utilities with previous work on ordinal utilities over continuous outcomes spaces (Chichilnisky, 1980, 1982a; Chichilnisky & Heal, 1983).

That this approach yields the same topological space of preferences as was found above can be seen as follows. Any utility function on M is determined by the values it assigns to the n vertices corresponding to the deterministic outcomes. Thus there is a natural correspondence between such functions and utility vectors $\mathbf{u} \in \mathbb{R}^n$ of the sort discussed in the previous section. Taking the gradient of such a function and then normalizing the result (when non-zero) corresponds to imposing equivalence under diagonal translation and positive scalar multiplication, respectively. Therefore the probability simplex approach also yields T^{n-2} as the space of utilities over n alternatives.

The equivalence between the two approaches also illustrates the dual nature of T^{n-2} as both the space of cardinal (i.e., interval scale) preferences over a finite set of alternatives and the space of linear ordinal preferences over all probabilistic mixtures of those alternatives. As Baryshnikov (2000) notes, ordinal linear preferences on $\mathbb{R}^m$ can be defined as equivalence classes in the dual of $\mathbb{R}^m$, with equivalence defined by positive scalar multiplication. Since this is the equivalence that defines ratio scale information, and $(\mathbb{R}^m)' \simeq \mathbb{R}^m$, such preferences can be identified with m-ary ratio scale observations. As in the probability simplex construction above (with $m = n-1$), when the null preference is allowed in Baryshnikov's framework the resulting space of preferences is T^{m-1}. Thus in addition to being the space of interval scale information, T can also be seen as the space of ratio scale information (on one less observation).

4. Consequences of null preference for aggregation

The present view of utilities, or cardinal preferences, as equivalence classes of real-valued functions shows that the null preference plays a unique role in the preference space, and that its inclusion changes the space's global properties (by making it contractible). A full treatment of preference aggregation must therefore consider all four possibilities according to whether or not null preference is allowed for individuals and for the social outcome. Therefore we now consider aggregation maps $f: P^k \to Q$, with $P, Q \in \{S^{n-2}, T^{n-2}\}$.[3]

An alternative, but equivalent, approach to this problem that does not rely on quotient spaces would be to impose an invariance property on aggregation maps, following D'Aspremont and Gevers (1977). Specifically, we could consider the original (pre-quotient) spaces $\mathbb{R}^n$ and $\mathbb{R}^n - \mathbb{R}\mathbf{1}$ and require that an aggregation map on these spaces be invariant under separate positive linear transformations of the inputs, up to positive linear transformation of the output. To that end, an aggregation map $g: U^k \to V$, with $U, V \in \{\mathbb{R}^n - \mathbb{R}\mathbf{1}, \mathbb{R}^n\}$, will be called *I-invariant* whenever $g(\mathbf{u}) \sim_I g(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in U^k$ satisfying $\forall j\, [\mathbf{u}^j \sim_I \mathbf{v}^j]$. (Here the superscript indexes voters; $\sim_I$ was defined in Section 3.1.) This requirement corresponds to the cardinal non-comparable (CN) axiom of D'Aspremont and Gevers (1977). We call g *I-anonymous* iff $g(\mathbf{u}^{\pi(1)}, \ldots, \mathbf{u}^{\pi(k)}) \sim_I g(\mathbf{u})$ for all $\mathbf{u} \in U^k$ and all bijections $\pi: \{1, \ldots, k\} \to \{1, \ldots, k\}$. Similarly, g is *I-unanimous* iff $\forall \mathbf{u} \in U\, [g(\mathbf{u}, \ldots, \mathbf{u}) \sim_I \mathbf{u}]$. Finally, the Pareto principle as applied to g is defined with respect to all pairs of convex combinations of the alternatives. We now show that, under these definitions, the question of the existence and

[3] To simplify notation, the superscript $n-2$ indicating the cardinality of the set of alternatives will often be suppressed. A superscript k will indicate the Cartesian product of a space, indexed over the set of k voters.

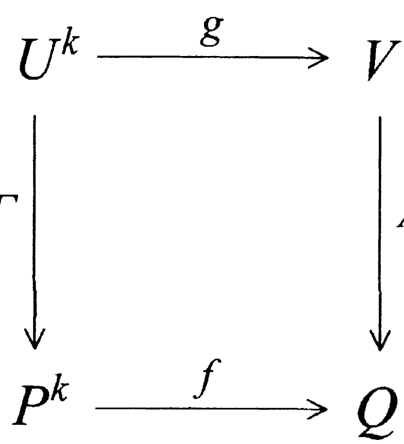

Fig. 2. Commutative diagram showing the equivalence between I-invariant maps $g: U^k \to V$ and maps on the quotient spaces $f: P^k \to Q$. U and V are both taken from $\{\mathbb{R}^n - \mathbb{R}\mathbf{1}, \mathbb{R}^n\}$, with P and Q their respective quotient spaces–S^{n-2} or T^{n-2}–under $\sim_I$. Γ and Λ are the corresponding quotient maps.

nature of aggregation maps $g: U^k \to V$ is equivalent to that for maps $f: P^k \to Q$ via the commutative diagram shown in Fig. 2.

Proposition 2. *Let* $U, V \in \{\mathbb{R}^n - \mathbb{R}\mathbf{1}, \mathbb{R}^n\}$ *with* $P = U/\sim_I$ *and* $Q = V/\sim_I$, *i.e.* $P, Q \in \{S^{n-2}, T^{n-2}\}$. *Let* $\Gamma: U^k \to P^k$ *and* $\Lambda: V \to Q$ *be the associated quotient maps* (Γ *is the k-fold product of the quotient map from U to P*). *For any continuous I-invariant* $g: U^k \to V$ *there exists a unique* $f: P^k \to Q$ *such that* $f \circ \Gamma = \Lambda \circ g$. *The converse is true except g is not unique. Furthermore, f is anonymous, unanimous, and/or Pareto if and only if g satisfies the corresponding combination of I-anonymity, I-unanimity, and Pareto.*

Proof. Given g, define f by $f(\mathbf{p}) = \Lambda(g(\mathbf{u}))$ for any $\mathbf{u} \in \Gamma^{-1}(\mathbf{p})$. This definition is consistent because g is I-invariant. For any $\mathbf{u} \in U^k$ we have $\mathbf{u} \in \Gamma^{-1}(\Gamma(\mathbf{u}))$ and thus $f(\Gamma(\mathbf{u})) = \Lambda(g(\mathbf{u}))$. Therefore $f \circ \Gamma = \Lambda \circ g$. If also $f' \circ \Gamma = \Lambda \circ g$ for some f', then given any $\mathbf{p} \in P^k$ and some $\mathbf{u} \in \Gamma^{-1}(\mathbf{p})$, $f'(\mathbf{p}) = f'(\Gamma(\mathbf{u})) = \Lambda(g(\mathbf{u})) = f(\mathbf{p})$. Hence $f' = f$ and f is unique. To see that f is continuous, let A be an arbitrary open set in Q and note that $\Gamma^{-1}(f^{-1}(A)) = (f \circ \Gamma)^{-1}(A) = (\Lambda \circ g)^{-1}(A)$. The latter set is open by continuity of Λ and g, so $f^{-1}(A)$ is open by definition of the quotient topology on P^k. (Note that the quotient topology on P^k is the same as the product topology derived from the quotient topology on P.) For the converse, observe that any continuous $f: P^k \to Q$ induces a map $f \circ \Gamma: U^k \to Q$. By the covering map lifting theorem as applied to Λ (see, e.g., Vick, 1994, Theorem 4.9), $f \circ \Gamma$ can be lifted to a continuous map $g: U^k \to V$ with $\Lambda \circ g = f \circ \Gamma$ provided that $(f \circ \Gamma)^*(\pi_1(U^k)) \subset \Lambda^*(\pi_1(V))$. This condition is trivially satisfied since in all cases $\Lambda^*: \pi_1(V) \to \pi_1(Q)$ is surjective (in fact $\pi_1(Q) = 0$ except when $Q = S^1$). Equivalence of anonymity, unanimity, and Pareto between f and g whenever g is I-invariant and $\Lambda \circ g = f \circ \Gamma$ can be verified by straightforward calculations. □

4.1. $T \times \cdots \times T \to T$

In the case where null preference is allowed both for individuals and for the society, there exist continuous, anonymous aggregation maps that respect unanimity. An example is the following map:

$$f(\mathbf{p}) = \begin{cases} \dfrac{\sum p_i}{\|\sum p_i\|} & \text{if } \sum p_i \neq 0 \;\&\; \forall i[p_i \neq 0], \\ 0 & \text{otherwise.} \end{cases}$$

This map returns the normalized average of all component preferences, where $T^{n-2} - \{0\} = S^{n-2}$ is embedded in $\mathbb{R}^n$ for the purposes of summation and scaling (see Section 3.1). The outcome is 0 whenever the profiles sum to 0 or any individual preference is 0. This function is easily seen to be anonymous, unanimous, and Pareto. However, the rule is still problematic, as indifference for any single voter nullifies the entire election. In fact, it can be shown that a version of this pathology will always arise, not just for Chichilnisky rules but for any continuous map from T^k to T.

Proposition 3. *Let* $f: T^k \to T$ *be continuous. Given any individual j and any profile* $\mathbf{p}_{-j}$ *for the remaining* $k-1$ *voters, define the component map* $\iota_j = f(\mathbf{p}_{-j}, \cdot): T \to T$. *Either* $\iota_j(0) = 0$ *or else* ι_j *is constant. In other words either the aggregated outcome is null if voter j has no strict preference, or else j has no influence whatsoever.*

Proof. Consider $q = f(p_{-j}, 0)$ and assume $q \neq 0$. For all $\epsilon > 0$ let $B_\epsilon(q)$ denote the open ϵ-ball in S around q (using the metric inherited from $S^{n-2} \subset \mathbb{R}^n$). The preimage $(\iota_j)^{-1}(B_\epsilon(q))$ contains 0 and is open in T (by continuity of ι_j), and thus must be equal to T. Therefore $Im(\iota_j) \subseteq B_\epsilon(q)$. Taking the intersection over all ϵ yields $Im(\iota_j) = \{q\}$. □

A better intuitive sense for this result can be achieved by noting the inherent instability in the null preference. An individual with such a preference can move to any one of the strict preferences via an infinitesimal perturbation; this fact is captured in the topology of T. Since continuity implies that small changes in the inputs yield small changes in the output, a voter's switch between 0 and any other preference must have an arbitrarily small effect on the social outcome. If this outcome lies in the sphere, whose topology is locally Euclidean, then the effect must be zero. Therefore whenever the rest of the society is able to come to a definitive conclusion without a strict opinion from the voter in question ($\iota_j(0)\neq 0$), the preference of that voter is irrelevant.

4.2. $T\times\cdots\times T\to S$

Proposition 4. *Any continuous map $f:T^k\to S$ must be constant.*

Proof. The argument is analogous to that for the proof of Proposition 3: Since in this case $f(\mathbf{0})\in S$, we have $\forall\varepsilon>0\,[Im(f)\subset B_\varepsilon(f(\mathbf{0}))]$ and thus $Im(f)=\{f(\mathbf{0})\}$. The only additional fact required for the proof to carry through is that T^k is the only open neighborhood of $\mathbf{0}$ in T^k. To see this, consider the basis for the topology on T^k consisting of sets of the form $A_1\times\cdots\times A_k$ with A_i open in T for all i. If $\mathbb{P}(A)$ denotes the property "$\mathbf{0}\notin A$ or $A=T^k$" then all members of the basis satisfy $\mathbb{P}$. Since $\mathbb{P}$ is clearly preserved by arbitrary unions and finite intersections, all open sets of T^n satisfy $\mathbb{P}$. □

Corollary. *There does not exist a continuous map $f:T^k\to S$ that respects unanimity (even ignoring the profile where all voters are indifferent).*

4.3. $S\times\cdots\times S\to S$

As reviewed already, the case where null preferences are not admitted has been well studied. Chichilnisky (1980) shows that there exist no Chichilnisky aggregators in this case. Chichilnisky (1982b) further shows that any map satisfying the WPA and Pareto conditions is homotopic to a dictatorship. Baryshnikov (2000) proves that this homotopy can be made a Pareto-isotopy, but that when WPA is dropped there exist Pareto aggregation functions that are not Pareto-isotopic to dictatorships.

4.4. $S\times\cdots\times S\to T$

The proof of the non-existence of Chichilnisky aggregators $f:S^k\to S$ relies on the degree of the aggregation map in the homotopy group $\pi_{n-2}(S^{n-2})$ (see Chichilnisky, 1980, 1996). Since T is contractible, $\forall m\,[\pi_m(T)=0]$, and such impossibility proofs break down. As it turns out, in the case of $S^k\to T$ there do exist Chichilnisky aggregation maps. One example is the averaging map:

$$f_A(\mathbf{p})=\begin{cases}\dfrac{\sum p_i}{\|\sum p_i\|} & \text{for } \sum p_i\neq 0,\\ 0 & \text{for } \sum p_i=0.\end{cases}$$

Here summation and scaling are based on the embedding of S^{n-2} into $\mathbb{R}^n$ as before. The map f_A is easily seen to satisfy every positive axiom defined here, namely continuity, anonymity, unanimity, efficiency, Pareto, and WPA. Chichilnisky (1982a) considers but rejects this map because under her disconnected topology it is not continuous. The map is also used by Le Breton and Uriarte (1990) in the context of the T-topology in their reply to Chichilnisky (1982a), as an example of how the choice of topology can critically affect the existence of aggregation maps. Le Breton and Uriarte (1990) also erroneously provide the averaging map as an example of an aggregator from T^k to T; however, the map is not continuous when 0 is included in the domain.

Contractibility of T implies that any two continuous maps into T from the same domain must be homotopic. In order to gain a more fine-grained picture of the space of maps $f:S^k\to T$ we apply Baryshnikov's (2000) isotopy approach and investigate when, for any two maps satisfying some combination of anonymity, unanimity, Pareto, and efficiency, there exists an isotopy that preserves that same set of properties. Equivalently, we can consider the subspaces of maps satisfying each possible combination of these axioms and ask which subspaces are connected in the homotopy topology. As the following proposition asserts, all such subspaces are connected, implying that even under the stricter requirements of isotopy all aggregation maps of any given sub-type are equivalent.

Proposition 5. *Let $f_0,f_1:S^k\to T$ be aggregation maps satisfying some combination of the axioms of anonymity, unanimity, Pareto, and efficiency. Then there exists a homotopy from f_0 to f_1 for which all intermediate maps satisfy all of the axioms assumed for f_0 and f_1.*

Proof. Define $\phi(\mathbf{p})=\sum_{(i,j)}\|p_i-p_j\|$, as a measure of the distance of any profile $\mathbf{p}$ from unanimity, and let $M=\max_{\mathbf{p}\in S^k}\phi(\mathbf{p})$. Assume first that f_0 and f_1 both respect unanimity, implying $f_0(\mathbf{p})=f_1(\mathbf{p})$ whenever $\phi(\mathbf{p})=0$. This fact, together with continuity of ϕ and $\|f_0-f_1\|:S^k\to\mathbb{R}$ and compactness of S^k, implies the existence of $\delta>0$ satisfying $\forall\mathbf{p}\,[\phi(\mathbf{p})<\delta\Rightarrow\|f_0(\mathbf{p})-f_1(\mathbf{p})\|<1]$. Note that if $\phi(\mathbf{p})<\delta$ then $\alpha f_0(\mathbf{p})+(1-\alpha)f_1(\mathbf{p})\neq 0$ for $\alpha\in[0,1]$. Now define the following homotopy from f_0 to f_1:

$$f_t(\mathbf{p})=\begin{cases} f_0(\mathbf{p}), & t<\frac{1}{4}\text{ or }\phi(\mathbf{p})>\max\\ & \{\delta,\delta+(4t-2)(M-\delta)\},\\ \frac{(2-4t)f_0(\mathbf{p})+(4t-1)f_1(\mathbf{p})}{\|(2-4t)f_0(\mathbf{p})+(4t-1)f_1(\mathbf{p})\|}, & \frac{1}{4}\leqslant t\leqslant\frac{1}{2},\phi(\mathbf{p})<\delta,\\ 0, & \phi(\mathbf{p})=\delta,\frac{1}{4}\leqslant t\leqslant\frac{1}{2},\\ 0, & \phi(\mathbf{p})=\delta+(4t-2)\\ & \times(M-\delta),\frac{1}{2}\leqslant t\leqslant\frac{3}{4},\\ f_1(\mathbf{p}), & \phi(\mathbf{p})<\delta+(4t-2)\\ & \times(M-\delta),t\geqslant\frac{1}{2}.\end{cases}$$

Fig. 3 gives a schematic description of this homotopy, which consists of two major phases. First, for all points $\mathbf{p}$ with $\phi(\mathbf{p})<\delta$, the outcome is moved directly from

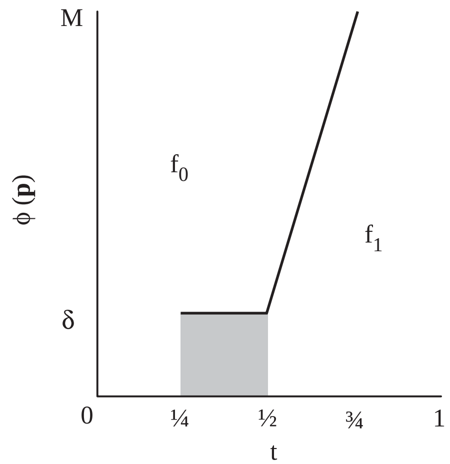

Fig. 3. Schematic of the homotopy $f_t(\mathbf{p})$, which is an anonymous-unanimous-efficient-Pareto-isotopy whenever the boundary maps satisfy all four of these axioms. The outcome is defined according to f_0 and f_1 in the regions respectively labeled by these maps. The result is null (0) along the solid line. The shaded region represents a continuous linear deformation from f_0 to f_1 in which outcomes are normalized convex combinations of the values of the two boundary functions.

$f_0(\mathbf{p})$ to $f_1(\mathbf{p})$ along the geodesic in S connecting these two outcomes. This deformation of the map is possible because the choice of δ ensures that all convex combinations of f_0 and f_1 in the region $\phi < \delta$ will avoid the center of the sphere. Second, a (measure-zero) boundary dividing a region mapped according to f_0 from one mapped according to f_1 sweeps across the profile space, starting at $\phi^{-1}(\delta)$. Points on the boundary are mapped to 0. The intermediate maps, which are gluings of f_0 and f_1 along this closed boundary, are continuous due to the non-Hausdorff topology of the outcome space T.

It is straightforward to verify that if both f_0 and f_1 are anonymous then all intermediate maps f_t are as well; similar statements hold for efficiency and Pareto. If either f_0 or f_1 fails to respect unanimity, an alternative homotopy can be defined that skips the first phase of the one described above; this can be achieved by taking $\delta = 0$ in the above homotopy formula. In this simplified homotopy, the intermediate maps satisfy each of anonymity, efficiency, and Pareto provided both f_0 and f_1 do. □

5. Discussion and conclusions

The choice of topology for the space of preferences can be critical in determining the existence and nature of aggregation functions, as argued by Le Breton and Uriarte (1990). Here we have derived the quotient topology on the space of utilities over a finite set of alternatives, under the assumption that utilities are only defined up to interval scale (von Neumann & Morgenstern, 1944). Under this topology, the null preference is in the same connected component as the rest of the space (as contrasted with previous approaches, e.g., Chichilnisky, 1985), and is in fact arbitrarily close to all other preferences. As a consequence, this space admits Chichilnisky aggregation functions. Our detailed analysis of the role of the null point showed further that there exist Chichilnisky functions if and only if null preference is allowed for the society. However, when individuals are allowed to be indifferent as well, a pathology arises whereby for all profiles, every individual is given either too much influence (the rest of society is unable to reach a strict preference without a strict preference from that voter) or else no influence at all. When null preference is allowed only for the society, simple and well behaved Chichilnisky rules exist; we therefore conclude that acceptable aggregation can only be achieved in this latter case.

The relationship among the results for the four scenarios considered here can be elucidated by noting that, in general, extending the range space of a class of maps (e.g., those defined by certain axioms) will relax the constraints on those maps and will thus enlarge the class. Similarly, extending the domain will add more constraint, as each map must be defined over a larger space. Therefore, it is not surprising that the only case that allows for well-behaved aggregation rules is the one ($S^k \to T$) that includes the null preference in the outcome space but not in the input space. Likewise, allowing indifference at the individual level but not the societal level (case $T^k \to S$) leads to the most restricted class of aggregation rules; only constant maps are possible. The two intermediate scenarios ($S^k \to S$ and $T^k \to T$), in which the choice and outcome spaces are equal, both lead to mixed results. For the case $S^k \to S$ there are no Chichilnisky rules, but there do exist maps more interesting than constant ones (e.g., dictatorships). For the case $T^k \to T$ there do exist Chichilnisky rules, but these (and all other maps) have the undesirable property of always over- or under-allocating power to any individual in any situation.

One might argue that our possibility results are trivially foreshadowed by Chichilnisky and Heal's (1983) resolution theorem, which maintains that Chichilnisky aggregation maps exist if and only if the preference space P is contractible. However, our results fall outside the scope of their theorem for two reasons. First, the resolution theorem only directly applies to situations in which the individual and social preference spaces are equal, and not to scenarios such as $f: S^k \to T$. Second, the space T, while contractible, does not have the structure of a parafinite CW complex (because it is not Hausdorff) and hence the resolution theorem, which depends on this technical assumption, does not apply. Thus our demonstration of a Chichilnisky aggregator for the case of $f : T^k \to T$, albeit an undesirable one, also constitutes a substantively new result. Nevertheless, the fact that our conclusion in this case coincides with the conclusion of Chichilnisky and Heal's theorem speaks to the power and generality of their result and suggests that the technical assumption required in their proof may be relaxable.

The approach taken here nearly fits into the framework that forms the conceptual basis of Chichilnisky and Heal's (1983) theorem, as the choice simplex is a special case of the outcome manifolds they consider. The critical difference concerns allowance of the null preference. When this option is removed, the space of linear ordinal preferences is homotopic to a sphere, which is non-contractible and thus does not admit a continuous, anonymous, unanimous aggregation function Chichilnisky (1980). Other work which has considered the null preference (Chichilnisky, 1982a, 1985) has used the Euclidean topology of $S \cup \{0\}$, i.e. with $\{0\}$ as an open set. Under this topology, inclusion of the indifference point has little effect, and the impossibility result holds. By contrast, the present derivations demonstrate that the proper topology (at least from the standpoint of utility theory) on the space of preferences including 0 is that given here in the definition of the topological space T. Here we have closed the gap resulting from this discrepancy by investigating T as preference space and presenting classification results concerning the aggregation maps that can arise depending upon when the null preference is allowed.

As mentioned previously, Le Breton and Uriarte (1990) describe the space T, with the same topology as derived here, in their demonstration of the dependence of impossibility results on the choice of preference topology. However, they go on to reject this topology as unsatisfactory because it is non-Hausdorff. They state the Hausdorff separation axiom as one of two requirements for any topology to be considered in analysis of social choice (the other requirement being meaningfulness), but give no specific reason for this statement. By way of reply, since the null preference is inherently unstable, requiring by definition perfect balance among all of the alternatives, any meaningful structure on preferences must place the 0 point arbitrarily close to all other preferences. This is precisely the result that arises here, when the topology is derived as above from the Euclidean topology on $\mathbb{R}^n$ in combination with the assumption of affine invariance. Therefore the non-Hausdorff topology is not only reasonable but natural for characterizing utility.

Our results touch upon a fundamental dilemma in social choice theory, namely the allowance of null social preferences as an acceptable outcome under certain symmetric voter profiles. Chichilnisky (1985) argues that it is unacceptable for any aggregation procedure to assign a null outcome in a situation where all voters have non-null preferences. The basis for her argument is that such a system allows for resolution of a Condorcet triple, i.e. a case of three voters with respective orderings of three alternatives given by (x, y, z), (y, z, x), and (z, x, y). The irresolvability of Condorcet triples is at the core of Arrow's (1963) proof, and without this fact his impossibility theorem would not hold. Similarly, Lauwers (2000) shows how difficulties with the Pareto principle in the topological framework reduce to cases where the voters are divided into two perfectly opposed factions, i.e. $\exists q \forall i\, [p_i \in \{q, -q\}]$. Continuity considerations imply that if the map is Pareto then the outcome must also come from $\{q, -q, 0\}$. Disallowing null outcomes forces an all-or-none choice between the wills of the two groups, which becomes the seed for dictatorship, as can be most easily seen in the case of two voters. In situations such as these (Condorcet triples and diametrical opposition), where the opinions of the voters as a group are perfectly symmetric with respect to the alternatives, common sense tells us that the only "fair" resolution is to declare a tie (or else to determine the outcome based on some stochastic mechanism such as the roll of a die). To the extent that ties present the danger of trivializing the problem, both the Arrovian and Chichilniskian social choice theories can be characterized as investigations into the existence of non-trivial, symmetry-breaking, yet socially acceptable solutions that completely avoid null outcomes. The negative results of these endeavors imply that insisting on a decisive choice in all cases will inevitably lead to inconsistencies. Hence both Arrow's (1963) impossibility theorem and the resolution theorem of Chichilnisky and Heal (1983) can be restated as that the requirement of a decisive outcome on all possible profiles is in direct conflict with our desire for well-behaved aggregation rules (i.e., anonymous and respecting unanimity, or following IR and non-dictatorship).

Viewed from this perspective our possibility result offers a nice compromise between competing goals, by allowing social preference to be fully indifferent only on a negligibly small subset of all possible voter profiles (including the outcome-symmetric ones discussed above). Our result on $f : T^k \to T$ further indicates that merely allowing for the possibility of null outcomes is insufficient to enable aggregation; the only way to achieve acceptable aggregation rules is to incorporate indifferent preferences at the social level but not at the individual level. The present results thus provide a complement to the impossibility theorems of Arrow (1963) and Chichilnisky (1980) by showing precisely when relaxing the constraint of a decisive outcome on a (zero measure) set of profiles can lead to successful resolution of the social choice paradox.

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