Lecture 11: t-test

Hypothesis Test for Population Mean
Goal: Infer $\mu$ from $M$
Null hypothesis: $\mu = \mu_0$
Usually 0
Sometimes another value, e.g. from larger population
Change scores
Memory improvement, weight loss, etc.
Sub-population within known, larger population
IQ of CU undergrads
Approach:
Determine likelihood function for $M$, using CLT
Compare actual sample mean to critical value

Likelihood Function for $M$
Probability distribution for $M$, according to $H_0$
Central Limit Theorem:
Mean equals population mean: $\mu_M = \mu_0$
Standard deviation: $\sigma_M = \frac{\sigma}{\sqrt{n}}$
Shape: Normal

Critical Value for $M$
Result that has 5% (a) chance of being exceeded, IF null hypothesis is true
Easier to find after standardizing $p(M)$
Convert distribution of sample means to z-scores: $z_M = \frac{M-\mu_M}{\sigma_M} = \frac{M-\mu_0}{\sigma/\sqrt{n}}$
Critical value for $z_M$ always the same
Just use $\text{qnorm}(\alpha)$
Equals 1.64 for $\alpha = 5$

Example: IQ
Are CU undergrads smarter than population?
Sample size $n = 100$, sample mean $M = 103$
Likelihood function
If no difference, what are probabilities of sample means?
Null hypothesis: $\mu_0 = 100$
CLT:
$\mu_M = 100$
$\sigma_M = s/\sqrt{n} = 15/10 = 1.5$
z-score: $\frac{(103-100)}{1.5} = 2$
Problem: Unknown Variance

\[ z_M = \frac{M - \mu_0}{\sigma / \sqrt{n}} \]

Test statistic depends on population parameter
   - Can only depend on data or values assumed by \( H_0 \)
Could include \( \sigma \) in null hypothesis
   - \( H_0: \mu = \mu_0 \) & \( \sigma = \sigma_0 \)
   - Usually no theoretical basis for choice of \( \sigma_0 \)
   - Cannot tell which assumption fails
Change test statistic
   - Replace population SD with sample SD
   - Depends only on data and \( \mu_0 \)
\[ t = \frac{M - \mu_0}{s / \sqrt{n}} \]

\( t \) Statistic
   - Invented in 1908 by “Student” at Guinness brewery
\[ t = \frac{M - \mu_0}{s / \sqrt{n}} \]
   - Deviation of sample mean divided by estimated standard error
   - Depends only on data and \( \mu_0 \)
   - Sampling distribution depends only on \( n \)

\( t \) Distribution
   - Sampling distribution of \( t \) statistic
   - Derived from ratio of Normal and (modified) \( \chi^2 \)
   - Depends only on sample size
   - Degrees of freedom: \( df = n - 1 \)
   - Invariant with respect to \( \mu, \sigma \)
   - Shaped like Normal, but with fatter tails
   - Reflects uncertainty in sample variance
   - Closer to Normal as \( n \) increases
   - Critical value decreases as \( n \) increases

\[ \alpha = .05 \]

<table>
<thead>
<tr>
<th>df</th>
<th>( t_{crit} )</th>
<th>df</th>
<th>( t_{crit} )</th>
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<tbody>
<tr>
<td>1</td>
<td>6.31</td>
<td>5</td>
<td>2.02</td>
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<tr>
<td>2</td>
<td>2.92</td>
<td>10</td>
<td>1.81</td>
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<tr>
<td>3</td>
<td>2.35</td>
<td>30</td>
<td>1.70</td>
</tr>
<tr>
<td>4</td>
<td>2.13</td>
<td>( \infty )</td>
<td>1.64</td>
</tr>
</tbody>
</table>
Steps of t-test
1. State clearly the two hypotheses
2. Determine null and alternative hypotheses
   \( H_0: \mu = \mu_0 \)
   \( H_1: \mu \neq \mu_0 \)
3. Compute the test statistic \( t \) from the data
   \[ t = \frac{M - \mu_0}{\sqrt{s/n}} \]
4. Determine likelihood function for test statistic according to \( H_0 \)
   \( t \) distribution with \( n-1 \) degrees of freedom
5. Find critical value
   \( R: \text{qt}(\alpha, n-1, \text{lower.tail=FALSE}) \)
6. Compare actual result to critical value
   \( t < t_{crit} \): Retain null hypothesis, \( \mu = \mu_0 \)
   \( t > t_{crit} \): Reject null hypothesis, \( \mu \neq \mu_0 \)

Example: Rat Mazes
Measure maze time on and off drug
   Difference score: \( \text{Time}_{\text{drug}} - \text{Time}_{\text{no drug}} \)
   Data (seconds): 5, 6, -8, -3, 7, -1, 1, 2
   Sample mean: \( M = 1.0 \)
   Sample standard deviation: \( s = 5.06 \)
   Standard error: \( \text{SE} = s/\sqrt{n} = 5.06/2.83 = 1.79 \)
   \( t = (M - \mu_0)/\text{SE} = (1.0 - 0)/1.79 = .56 \)
   Critical value: \( \text{qnorm}(.05, 7, \text{lower.tail=FALSE}) = 1.89 \)
   \( t \) does not exceed critical value
   Cannot reject null hypothesis
   Assume no effect of drug